Green cells below match corresponding answers in the text.

1 - 5 Legendre polynomials and functions

5. Obtain P_6 and P_7.

```
ClearAll["Global`*"]
```

It turns out that Legendre polynomials are available from a built-in command.

```
LegendreP[6, x]
```

$$\frac{1}{16}\left(-5 + 105\, x^2 - 315\, x^4 + 231\, x^6\right)$$

```
LegendreP[7, x]
```

$$\frac{1}{16}\left(-35\, x + 315\, x^3 - 693\, x^5 + 429\, x^7\right)$$

11 - 15 Further formulas

11. ODE. Find a solution of $\left(a^2 - x^2\right) y'' - 2\, x\, y' + n\,(n + 1)\, y = 0,\ \ a \neq 0,$ by reduction to the Legendre equation.

```
ClearAll["Global`*"]
```

```
eqn = (a^2 - x^2) y''[x] - 2 x y'[x] + n (n + 1) y[x] == 0
```

$$n\,(1 + n)\, y[x] - 2\, x\, y'[x] + \left(a^2 - x^2\right) y''[x] == 0$$

```
sol = DSolve[eqn, y, x, Assumptions → a ≠ 0]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ C[1]\ \text{LegendreP}\left[n,\ \frac{x}{a}\right] + C[2]\ \text{LegendreQ}\left[n,\ \frac{x}{a}\right]\right]\right\}\right\}$$

```
sol1 = sol /. {C[1] → 1, C[2] → 1, n → 1, a → 1}
```

$$\left\{\left\{y \to \text{Function}\left[\{x\},\ 1\ \text{LegendreP}\left[1,\ \frac{1}{1}\, x\right] + 1\ \text{LegendreQ}\left[1,\ \frac{1}{1}\, x\right]\right]\right\}\right\}$$

$$\text{LegendreP}\left[1,\ \frac{1}{1}\, x\right] + 1\ \text{LegendreQ}\left[1,\ \frac{1}{1}\, x\right]$$

$$-1 + x + x\left(-\frac{1}{2}\,\text{Log}[1 - x] + \frac{1}{2}\,\text{Log}[1 + x]\right)$$

15. Associated Legendre functions $P_n^k[x]$ are needed, e.g. in quantum physics. They are defined by $P_n^k[x] = \left(1 - x^2\right)^{k/2} \frac{d^k p_n[x]}{d x^k}$ and are solutions of the ODE

$(1-x^2)\,y'' - 2\,x\,y' + q[x]\,y = 0$ where $q[x] = n(n+1) - k^2 / (1-x^2)$. Find P_1^1[x], P_2^1[x], P_2^2[x], and P_4^2[x] and verify that they satisfy numbered line (16) in yellow above.

$$P_1^1[x] = (1-x^2)^{1/2} \frac{d^1 p_1[x]}{d\,x^1}$$

```
LegendreP[1, x]
```

x

$$P_1^1[x] = (1-x^2)^{1/2}$$

$$P_2^1[x] = (1-x^2)^{1/2} \frac{d^1 p_2[x]}{d\,x^1}$$

```
LegendreP[2, x]
```

$$\frac{1}{2}(-1+3\,x^2)$$

```
D[%, x]
```

3 x

$$(1-x^2)^{1/2} * \%$$

$$3\,x\,\sqrt{1-x^2}$$

$$P_2^2[x] = (1-x^2)^{2/2} \frac{d^2 p_2[x]}{d\,x^2}$$

```
LegendreP[2, x]
```

$$\frac{1}{2}(-1+3\,x^2)$$

```
D[%, {x, 2}]
```

3

$$3\,(1-x^2)$$

$$P_4^2[x] = (1-x^2)^{2/2} \frac{d^2 p_4[x]}{d\,x^2}$$

```
LegendreP[4, x]
```

$$\frac{1}{8}(3-30\,x^2+35\,x^4)$$

```
D[%, {x, 2}]
```

$$\frac{1}{8}\left(-60 + 420\, x^2\right)$$

```
FullSimplify[% * (1 - x^2)]
```

$$-\frac{15}{2}\left(1 - 8\, x^2 + 7\, x^4\right)$$

```
PossibleZeroQ[-15/2 (1 - 8 x^2 + 7 x^4) - ((1 - x^2) (105 x^2 - 15) / 2)]
```

$$\text{PossibleZeroQ}\left[-\frac{15}{2}\left(1 - 8\, x^2 + 7\, x^4\right) - \left(\left(1 - x^2\right)\left(105\, x^2 - 15\right)/2\right)\right]$$

True

The green cells above match the answers in the text. (The last, though slightly different in format, is verified by the PZQ.)